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Global blow-up for a localized nonlinear parabolic equation with a nonlocal boundary condition ☆

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ABSTRACT

This paper deals with the blow-up properties of positive solutions to a nonlinear parabolic equation with a localized reaction source and a nonlocal boundary condition. Under certain conditions, the blowup criteria is established. Furthermore, when $f(u) = u^p$, $0 < p \leqslant 1$, the global blowup behavior is shown, and the blowup rate estimates are also obtained.

1. Introduction

In this paper we consider the following nonlinear parabolic equation with a localized reaction source and a weighted nonlocal boundary condition

$$\begin{aligned}
&u_t = f(u)\big(\Delta u + au(x_0,t)\big), \quad x \in \Omega,\ t > 0,\\
&u(x,t) = \int_\Omega g(x,y)u(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0,\\
&u(x,0) = u_0(x), \quad x \in \Omega,
\end{aligned} \tag{1.1}$$

where a is a positive constant, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$ and $x_0 \in \Omega$ is a fixed point.

Problem (1.1) arises in the study of the flow of a fluid through a porous medium with an internal localized source and in the study of population dynamics (see [1,8–10,2,13]). There has been a considerable amount of literature dealing with the properties of solutions to local semilinear parabolic equations or systems of heat equations with homogeneous Dirichlet boundary conditions or with nonlinear boundary conditions (see [25,14,16,22,11,24,18] and references therein). However, there are some important phenomena formulated as parabolic equations which are coupled with nonlocal boundary condi-

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tions in mathematical modeling such as thermoelasticity theory (see[5,6]). In this case, the solution $u(x,t)$ describes entropy per volume of the material. The problem of nonlocal boundary value for linear parabolic equations of the type

$$\begin{aligned}
&u_t - Au = c(x)u, \quad x \in \Omega,\ t > 0,\\
&u(x,t) = \int_\Omega K(x,y)u(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0,\\
&u(x,0) = u_0(x), \quad x \in \Omega
\end{aligned} \tag{1.2}$$

with uniformly elliptic operator $A = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i}$ and $c(x) \leqslant 0$ was studied by Friedman [15]. The global existence and monotonic decay of the solution of problem (1.2) were obtained under the condition $\int_\Omega |k(x,y)|\,dy < 1$ for all $x \in \partial\Omega$. And later the problem (1.2) with Au replaced by Δu and the linear term $c(x)u$ replaced by the nonlinear term $g(x,u)$ was discussed by Deng [7]. The comparison principle and the local existence were established. On the basis of Deng's work, Seo in [23] investigated the above problem with $g(x,u) = g(u)$, by using the upper and lower solutions' technique, he gained the blowup condition of the positive solution, and in the special case $g(u) = u^p$ or $g(u) = e^u$ he also derived the blowup rate estimates.

As for more general discussions on the dynamics of parabolic problem with nonlocal boundary condition, one can see, e.g. [19,20] by Pao, where the following problem

$$\begin{aligned}
&u_t - Au = g(x,u), \quad x \in \Omega,\ t > 0,\\
&\alpha_0\frac{\partial u(x,t)}{\partial \nu} + u(x,t) = \int_\Omega K(x,y)u(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0,\\
&u(x,0) = u_0(x), \quad x \in \Omega
\end{aligned} \tag{1.3}$$

was considered and recently in [21] Pao gave the numerical solutions of diffusion equations with nonlocal boundary conditions.

Parabolic equations with both nonlocal sources and nonlocal boundary conditions have been studied as well. For example, the problem of the form

$$\begin{aligned}
&u_t - \Delta u = \int_\Omega g(u)\,dx, \quad x \in \Omega,\ t > 0,\\
&u(x,t) = \int_\Omega K(x,y)u(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0,\\
&u(x,0) = u_0(x), \quad x \in \Omega
\end{aligned} \tag{1.4}$$

was studied by Lin and Liu [17]. They established local existence, global existence and nonexistence of solutions and discussed the blow-up properties of solutions.

Recently, porous medium equations with local sources or with nonlocal sources subjected to nonlocal boundary conditions were studied by Wang et al. [26] and by Cui et al. [4]. And the blow-up conditions and the blow-up rate estimates were obtained.

The above studies show that the growth or decay properties of the solutions to above problems depend on the growth of the nonlinear reaction term $g(u)$, which is similar to general semilinear equations with homogeneous Dirichlet boundary conditions. On the other hand, due to the appearance of the nonlocal boundary condition, the properties of the solutions heavily depend on the kernel $K(x,y)$ as well.

Motivated by the above works, we are interested in the blow-up properties of problem (1.1). The aim of this paper is twofold. Firstly, we establish the global existence and finite time blow-up of the solution of problem (1.1). Secondly, we discuss the blow-up profile for special case of $f(u)$.

Before stating our main results, we make some assumptions on $f(s)$, the kernel $g(x,y)$ and the initial datum $u_0(x)$ as follows:

(H_1) $f(s) \in C([0,\infty)) \cap C^1(0,\infty)$ such that $f(0) \geqslant 0$ and $f'(s) > 0$ in $(0,\infty)$.
(H_2) $g(x,y)$ is continuous and nonnegative on $\partial\Omega \times \overline{\Omega}$ with $\int_\Omega g(x,y)\,dy > 0$ for all $x \in \partial\Omega$.
(H_3) $u_0 \in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, $u_0(x) > 0$ in Ω, $u_0(x) = \int_\Omega g(x,y)u_0(y)\,dy$ on $\partial\Omega$.

Our main results read as follows.

Theorem 1.1. *Suppose that $f(s)$, $g(x,y)$ and $u_0(x)$ satisfy (H_1)–(H_3), that $\int_\Omega g(x,y)\,dy \geqslant 1$ on $\partial\Omega$ and that $\int_\delta^{+\infty} \frac{ds}{f(s)s} < +\infty$ for some positive constant δ, then the solution $u(x,t)$ of problem (1.1) blows up in finite time.*

Theorem 1.2. *Let hypothesis* (H$_2$) *holds and assume that* $\int_\Omega g(x,y)\,dy < 1$ *on* $\partial\Omega$, *then there exists a unique positive solution* $\psi(x)$ *to the following elliptic problem*:

$$-\Delta\psi(x) = 1, \quad x \in \Omega; \qquad \psi(x) = \int_\Omega g(x,y)\psi(y)\,dy, \quad x \in \partial\Omega. \tag{1.5}$$

Theorem 1.3. *Let hypotheses* (H$_1$)–(H$_3$) *hold, and assume that* $\int_\Omega g(x,y)\,dy < 1$ *on* $\partial\Omega$, *then all solutions of problem* (1.1) *are global under either one of the following two conditions*:

(i) $a\psi(x_0) \leqslant 1$;
(ii) $\int_\delta^{+\infty} \frac{ds}{f(s)s} = +\infty$ *for some constant* $\delta > 0$.

Theorem 1.4. *Let hypotheses* (H$_1$)–(H$_3$) *hold, and assume that* $\int_\Omega g(x,y)\,dy < 1$ *on* $\partial\Omega$, *then every solution of problem* (1.1) *blows up in finite time if* $a\psi(x_0) > 1$ *and* $\int_\delta^{+\infty} \frac{ds}{f(s)s} < +\infty$ *for some* $\delta > 0$.

To describe the blow-up profile of the blowup solutions, we need the following two additional assumptions on the initial datum $u_0(x)$:

(H$_4$) $\Delta u_0(x) \leqslant 0$ and $\Delta u_0(x) + au_0(x_0) \geqslant 0$ on $\overline{\Omega}$.

And then we have:

Theorem 1.5. *Let hypotheses* (H$_1$)–(H$_4$) *hold, and let* $f(u) = u^p$, $0 < p \leqslant 1$. *If* $\int_\Omega g(x,y)\,dy < 1$ *on* $\partial\Omega$ *and the solution* $u(x,t)$ *of problem* (1.1) *blows up in finite time, then the blow-up set of* $u(x,t)$ *is the whole domain* Ω. *Furthermore, if we denote the blow-up time of* $u(x,t)$ *by* T^*, *then for the case* $0 < p < 1$ *there exist three positive constants* d, D *and* D' *such that*

$$d\big(T^* - t\big)^{-\frac{1}{p}} \leqslant \max_{x\in\overline{\Omega}} u(x,t) \leqslant D\big(T^* - t\big)^{-\frac{1}{p}} + D'. \tag{1.6}$$

This paper is organized as follows. In Section 2, we show the comparison principle and the local existence. In Section 3, some criteria for the positive solution to exist globally or to blow up in finite time is given. In Section 4, the global blow-up result and the blow-up rate estimates of blow-up solutions for the special case of $f(s)$ are obtained.

2. The comparison principle and the local existence

In this section we start with the definition of supersolution and subsolution of problem (1.1). For convenience, we set $Q_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times (0,T]$, $Q_t = \Omega \times (0,t]$, $0 < t \leqslant T < +\infty$, and $\overline{Q}_T$, $\overline{Q}_t$ be their respective closures.

Definition 2.1. A function $\hat{u}(x,t)$ is called a subsolution of problem (1.1) in Q_T, if $\hat{u}(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies

$$\begin{aligned} &\hat{u}_t \leqslant f(\hat{u})\big(\Delta\hat{u} + a\hat{u}(x_0,t)\big), \quad (x,t) \in Q_T, \\ &\hat{u}(x,t) \leqslant \int_\Omega g(x,y)\hat{u}(y,t)\,dy, \quad (x,t) \in S_T, \\ &\hat{u}(x,0) \leqslant u_0(x), \quad x \in \Omega. \end{aligned} \tag{2.1}$$

A supersolution $\tilde{u}(x,t)$ of problem (1.1) is defined analogously by the above inequalities with each inequality reversed. A solution of problem (1.1) is a function which is both a subsolution and a supersolution of problem (1.1).

Before studying our problem, we give a comparison lemma.

Lemma 2.2. *Assume that* $w(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ *and satisfies*

$$\begin{aligned} &w_t - d(x,t)\Delta w \geqslant \sum_{i=1}^n b_i(x,t)w_{x_i} + c_1(x,t)w + c_2(x,t)w(x_0,t), \quad (x,t) \in Q_T, \\ &w(x,t) \geqslant c_3(x,t)\int_\Omega c_4(x,y)w(y,t)\,dy, \quad (x,t) \in S_T, \\ &w(x,0) > 0, \quad x \in \overline{\Omega}, \end{aligned} \tag{2.2}$$

where $d(x,t)$, $b_i(x,t)$ $(i=1,2,\dots,n)$ and $c_j(x,t)$ $(j=1,2,3)$ are continuous in Q_T, $c_1(x,t)$, $c_2(x,t)$ are bounded in Q_T, $d(x,t) \geqslant d_0 > 0$, $c_2(x,t), c_3(x,t) \geqslant 0$ in Q_T and $c_4(x,y)$ is nonnegative and continuous on $\partial\Omega \times \Omega$ and is not identically zero. Then $w(x,t) \geqslant 0$ on $\overline{Q}_T$.

The proof is a trivial modification of that of Theorem 2.1 in [7] or of Lemma 2.1 of [3]. We omit it here.

Remark 2.3. If $c_3(x,t)\int_\Omega c_4(x,y)\,dy \leqslant 1$ on S_T and $w(x,t)$ satisfies all inequalities in (2.2) except with the third inequality replaced by $w(x,0) \geqslant 0$ on $\overline{\Omega}$, then also we have $w(x,t) \geqslant 0$ on $\overline{Q}_T$.

In order to get the global existence and finite time blow-up results for problem (1.1), we need yet the following comparison principle which is a direct consequence of Lemma 2.2 and Remark 2.3.

Lemma 2.4. *Assume that $\tilde u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is a nonnegative supersolution of problem (1.1) and $\hat u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ is a nonnegative subsolution of problem (1.1), that there exists a small positive constant η such that $\tilde u(x,t) \geqslant \eta$ on $\overline{Q}_T$ or $\hat u(x,t) \geqslant \eta$ on $\overline{Q}_T$, and that $\tilde u(x,0) > \hat u(x,0)$ on $\overline{\Omega}$ or $\tilde u(x,0) \geqslant \hat u(x,0)$ on $\overline{\Omega}$ if $\int_\Omega g(x,y)\,dy \leqslant 1$ on $\partial\Omega$. Then $\tilde u(x,t) \geqslant \hat u(x,t)$ on $\overline{Q}_T$.*

Local in time existence of the positive classical solution of problem (1.1) can be obtained by using fixed point theorem, the representation formula and the contraction mapping principle as in [27,17]. By the above comparison principle, we can get the uniqueness of the solution to problem (1.1), and then we have:

Theorem 2.5. *Let hypotheses (H_1)–(H_3) hold, then there exist T^* $(0 < T^* \leqslant +\infty)$ and $u(x,t) \in C(\overline{\Omega} \times [0,T^*)) \cap C^{2,1}(\Omega \times (0,T^*))$, such that $u(x,t)$ is the unique maximal in time solution of problem (1.1). If $T^* < +\infty$, then we have $\limsup_{t\to T^*} \sup_{x\in\Omega} u(x,t) = +\infty$.*

The proof is more or less standard, and is therefore omitted here.

3. The blow-up criteria

In this section we give out the proofs of Theorems 1.1–1.4. Comparing with usual homogeneous Dirichlet boundary condition, we can find out that the kernel $g(x,y)$ plays an important role in the global existence and global nonexistence for problem (1.1).

Proof of Theorem 1.1. In virtue of hypotheses (H_3) and (H_2), we know that $u_0(x) > 0$ on $\overline{\Omega}$. Then we can choose a constant v_0 such that $0 < v_0 < \min_{x\in\overline{\Omega}} u_0(x)$ and consider the initial value problem of the following ordinary differential equation

$$\begin{aligned} &v'(t) = af(v)v, \quad t > 0, \\ &v(0) = v_0. \end{aligned} \tag{3.1}$$

From hypothesis (H_1) and the theory of ordinary differential equations, we know that there exists a unique solution $v(t)$ of problem (3.1) which increases in the time variable t. Since $\int_\delta^{+\infty} \frac{ds}{f(s)s} < +\infty$ for some positive constant δ, $v(t)$ blows up at finite time $T_v^* = \frac{1}{a}\int_{v_0}^{+\infty} \frac{ds}{f(s)s} < +\infty$. Due to the condition $\int_\Omega g(x,y)\,dy \geqslant 1$, we can easily verify that the solution $v(t)$ of problem (3.1) is a subsolution of problem (1.1). Noting that $v(t) \geqslant v_0 > 0$ and $u_0(x) > v(0)$ on $\overline{\Omega}$, by using the comparison principle Lemma 2.4, we know that $u(x,t) \geqslant v(t)$ for $x \in \overline{\Omega}$, $t > 0$, and this shows that the solution $u(x,t)$ of problem (1.1) blows up in finite time. □

From now on, we begin to study problem (1.1) in the case $\int_\Omega g(x,y)\,dy < 1$ on $\partial\Omega$. First, we consider the linear elliptic problem (1.5), that is,

$$\begin{aligned} &-\Delta\psi(x) = 1, \quad x \in \Omega, \\ &\psi(x) = \int_\Omega g(x,y)\psi(y)\,dy, \quad x \in \partial\Omega, \end{aligned}$$

and give out the proof of Theorem 1.2.

Proof of Theorem 1.2. Choose $\psi_0(x) = u_0(x)$ on $\overline{\Omega}$ and define a sequence $\{\psi_m(x)\}$ inductively as follows: for given $\psi_m(x)$, let $\tilde\psi_{m+1}(x) = \int_\Omega g(x,y)\psi_m(y)\,dy$, $x \in \partial\Omega$, and let $\psi_{m+1}(x)$ be the solution of the following linear elliptic problem

$$\begin{aligned} &-\Delta\psi_{m+1}(x) = 1, \quad x \in \Omega, \\ &\psi_{m+1}(x) = \tilde\psi_{m+1}(x), \quad x \in \partial\Omega. \end{aligned} \tag{3.2}$$

By the theory of linear elliptic equations, we know that $\psi_{m+1}(x)$ exists, and is positive and continuous on $\overline{\Omega}$ provided the same is true for $\psi_m(x)$. Further, by using (3.2), we have

$$-\Delta\big(\psi_{m+1}(x)-\psi_m(x)\big)=0, x\in\Omega. \tag{3.3}$$

Then the maximum principle of elliptic equations [12, Chapter 2] implies

$$\sup_{x\in\overline{\Omega}}\big|\psi_{m+1}(x)-\psi_m(x)\big|=\sup_{x\in\partial\Omega}\big|\psi_{m+1}(x)-\psi_m(x)\big|.$$

In virtue of (H_2), $g(x,y)$ is a nonnegative continuous function on $\partial\Omega\times\Omega$, then from $\int_\Omega g(x,y)\,dy<1$, we know that $\rho=\max_{x\in\partial\Omega}\int_\Omega g(x,y)\,dy<1$. And then by induction, we obtain

$$\sup_{x\in\overline{\Omega}}\big|\psi_{m+1}(x)-\psi_m(x)\big|\leqslant C\rho^m, \tag{3.4}$$

where $C=\sup_{x\in\overline{\Omega}}|\psi_1(x)-\psi_0(x)|$, and Eq. (3.4) shows that $\{\psi_m(x)\}$ is a uniform Cauchy sequence in $C(\overline{\Omega})$. By standard theory of elliptic equations (see also [12, Chapter 2]), we know that $\psi(x)=\lim_{m\to+\infty}\psi_m(x)$ is a solution of problem (1.5) and $\psi(x)\in C(\overline{\Omega})\cap C^2(\Omega)$.

Finally, if $\varphi(x)$ is another solution of problem (1.5), then we have

$$\begin{aligned}&-\Delta\big(\psi(x)-\varphi(x)\big)=0,\quad x\in\Omega,\\&\psi(x)-\varphi(x)=\int_\Omega g(x,y)\big(\psi(y)-\varphi(y)\big)\,dy,\quad x\in\partial\Omega.\end{aligned}$$

Again by the elliptic maximum principle, we have

$$\begin{aligned}\sup_{x\in\overline{\Omega}}\big|\psi(x)-\varphi(x)\big|&=\sup_{x\in\partial\Omega}\big|\psi(x)-\varphi(x)\big|\\&=\sup_{x\in\partial\Omega}\left|\int_\Omega g(x,y)\big(\psi(y)-\varphi(y)\big)\,dy\right|\\&\leqslant\rho\sup_{x\in\overline{\Omega}}\big|\psi(x)-\varphi(x)\big|.\end{aligned} \tag{3.5}$$

Since $\rho<1$, (3.5) implies that $\varphi(x)\equiv\psi(x)$, then the uniqueness of the positive solution of problem (1.5) follows. And then we complete the proof of Theorem 1.2. □

Now, we can show the global existence and global nonexistence.

Proof of Theorem 1.3. (i) Let $\psi(x)$ be the unique positive solution of the linear elliptic problem (1.5), from the proof of Theorem 1.2 we know that $\psi(x)\geqslant 0$ on $\overline{\Omega}$, then the elliptic maximum principle and hypothesis (H_2) ensure that $\psi(x)>0$ on $\overline{\Omega}$. And let $\max_{x\in\overline{\Omega}}\psi(x)=K_1$, $\min_{x\in\overline{\Omega}}\psi(x)=K_2$, then $K_1,K_2>0$. We define a function $w(x,t)$ as follows:

$$w(x,t)=M\psi(x), \tag{3.6}$$

where M is a constant to be determined later. Noting that $a\psi(x_0)\leqslant 1$, we have for $x\in\Omega$, $t>0$,

$$w_t-f(w)\big(\Delta w+aw(x_0,t)\big)=f\big(M\psi(x)\big)\big(-M\Delta\psi(x)-aM\psi(x_0)\big)=f\big(M\psi(x)\big)M\big(1-a\psi(x_0)\big)\geqslant 0. \tag{3.7}$$

On the other hand, by using the fact that $\psi(x)$ is the solution of problem (1.5), we have for $x\in\partial\Omega$,

$$w(x,t)=M\psi(x)=M\int_\Omega g(x,y)\psi(y)\,dy=\int_\Omega g(x,y)w(y,t)\,dy. \tag{3.8}$$

Choose $M>K_2^{-1}\max_{x\in\overline{\Omega}}u_0(x)$, then $w(x,0)=M\psi(x)\geqslant MK_2>u_0(x)$ on $\overline{\Omega}$. Combing this inequality with (3.7) and (3.8), we know that $w(x,t)$ defined as (3.6) is a supersolution of problem (1.1). Since $w(x,t)\geqslant MK_2>0$, $w(x,0)>u_0(x)$, and $w(x,t)$ exists globally, by Lemma 2.4, we know that $u(x,t)\leqslant w(x,t)$. And then $u(x,t)$ exists globally.

(ii) Choose $b>a$ and $z_0>\max_{x\in\overline{\Omega}}u_0(x)$, and consider the following initial value problem

$$\begin{aligned}&z'(t)=bf\big(z(t)\big)z(t),\quad t>0,\\&z(0)=z_0.\end{aligned} \tag{3.9}$$

It follows from hypothesis (H_1) and the theory of ordinary differential equations that there exists a unique solution $z(t)$ to problem (3.9) and $z(t)$ is increasing. Noticing the condition $\int_\delta^{+\infty} \frac{ds}{f(s)s} = +\infty$ for some $\delta > 0$, we also know that the solution $z(t)$ of problem (3.9) exists globally. Set $w(x,t) = z(t)$, then by using the condition $\int_\Omega g(x,y)\,dy < 1$ on $\partial\Omega$, we obtain

$$\begin{aligned}
&w_t - f(w)\big(\Delta w + a w(x_0,t)\big) = z'(t) - f\big(z(t)\big)\big(\Delta z(t) + a z(t)\big) = (b-a) f\big(z(t)\big) z(t) > 0, \quad x \in \Omega,\ t > 0, \\
&w(x,t) = z(t) > \int_\Omega g(x,y) z(t)\,dy = \int_\Omega g(x,y) w(x,t)\,dy, \quad x \in \partial\Omega,\ t > 0, \\
&w(x,0) = z(0) = z_0 > u_0(x), \quad x \in \overline{\Omega}.
\end{aligned} \tag{3.10}$$

The above inequalities show that $w(x,t) = z(t)$ is the supersolution of problem (1.1), noting that $w(x,t) = z(t) > z_0 > 0$, then the comparison principle Lemma 2.4 implies that the solution of problem (1.1), $u(x,t) \leqslant w(x,t)$. And then $u(x,t)$ exists globally.

From (i) and (ii), we complete the proof of Theorem 1.3. □

Proof of Theorem 1.4. Since $a\psi(x_0) > 1$, we have $b_1 = K_1^{-1}(a\psi(x_0) - 1) > 0$, where K_1, K_2 are positive constants represent the maximum and minimum of the solution $\psi(x)$ of problem (1.5) on $\overline{\Omega}$. From the proof of Theorem 1.1 we know that the initial datum $u_0(x) > 0$ on $\overline{\Omega}$. Let $z(t)$ be the solution of the following initial value problem of ordinary differential equation,

$$\begin{aligned}
&z'(t) = b_1 f\big(K_2 z(t)\big) z(t), \quad t > 0, \\
&z(0) = z_0,
\end{aligned} \tag{3.11}$$

where $0 < z_0 < K_1^{-1} \min_{x\in\overline{\Omega}} u_0(x)$. Then $z(t)$ is increasing and $z(t) \geqslant z_0 > 0$. Due to the condition $\int_\delta^{+\infty} \frac{ds}{f(s)s} < +\infty$ for some $\delta > 0$, we know that the solution $z(t)$ of problem (3.11) blows up in finite time.

Set $w(x,t) = z(t)\psi(x)$, then for $x \in \Omega$, $t > 0$, we have

$$\begin{aligned}
w_t - f(w)\big(\Delta w + a w(x_0,t)\big) &= z'(t)\psi(x) - f\big(z(t)\psi(x)\big)\big(z(t)\Delta\psi(x) + a z(t)\psi(x_0)\big) \\
&\leqslant z'(t) K_1 - f\big(K_2 z(t)\big) z(t)\big(a\psi(x_0) - 1\big) = 0.
\end{aligned} \tag{3.12}$$

On the other hand, for $x \in \partial\Omega$, $t > 0$, we have

$$w(x,t) = z(t)\psi(x) = z(t)\int_\Omega g(x,y)\psi(y)\,dy = \int_\Omega g(x,y) w(y,t)\,dy. \tag{3.13}$$

Also for $x \in \overline{\Omega}$, we have

$$w(x,0) = z(0)\psi(x) = z_0\psi(x) < K_1^{-1}\psi(x)\min_{x\in\overline{\Omega}} u_0(x) \leqslant u_0(x). \tag{3.14}$$

And the inequalities (3.12)–(3.14) show that $w(x,t)$ is a subsolution of problem (1.1). Since $w(x,t) = z(t)\psi(x) \geqslant z_0 K_2 > 0$ and $w(x,t)$ blows up in finite time, Lemma 2.4 implies that the solution $u(x,t)$ of problem (1.1) satisfies $u(x,t) \geqslant w(x,t)$. Then $u(x,t)$ blows up in finite time, and this completes the proof of Theorem 1.4. □

4. Blow-up profile

In this section we give out the proof of Theorem 1.5. Throughout this section we assume that $f(u) = u^p$, $0 < p \leqslant 1$ and that the solution $u(x,t)$ of problem (1.1) blows up in finite time. It is easily to verify that $f(s)$ satisfies hypothesis (H_1). We denote by T^* the blow-up time of the blowup solution $u(x,t)$ of problem (1.1) and set

$$h(t) = a u(x_0,t), \qquad H(t) = \int_0^t h(s)\,ds. \tag{4.1}$$

We divide the proof of Theorem 1.5 into the following several lemmas.

Lemma 4.1. *Let hypotheses* (H_2)–(H_4) *hold, assume that* $\int_\Omega g(x,y)\,dy < 1$ *for* $x \in \partial\Omega$ *and that* $u(x,t)$ *is the solution of problem* (1.1). *Then* $\Delta u \leqslant 0$ *in* $\Omega \times (0,T^*)$.

Proof. Differentiating equation (1.1) with respect to t, from the condition $\Delta u_0(x) + a u_0(x_0) \geqslant 0$ on $\overline{\Omega}$ in (H_4), Lemma 2.2 and Remark 2.3, we can easily obtain $u_t(x,t) \geqslant 0$ on $\overline{\Omega} \times [0,T^*)$, and then we know that

$$u(x,t) \geqslant u_0(x) \geqslant \min_{x\in\overline{\Omega}} u_0(x) (= \eta) > 0. \tag{4.2}$$

Let $w(x,t)=\Delta u(x,t)$, then it follows from (1.1) that for $(x,t)\in\Omega\times(0,T^*)$,

$$\begin{aligned} w_t &= u^p\Delta w+2pu^{p-1}\nabla u\cdot\nabla w+pu^{p-1}\big(\Delta u+au(x_0,t)\big)w+p(p-1)u^{p-2}\big(\Delta u+au(x_0,t)\big)|\nabla u|^2 \\ &= u^p\Delta w+2pu^{p-1}\nabla u\cdot\nabla w+pu^{-1}u_t w+p(p-1)u^{-2}u_t|\nabla u|^2. \end{aligned}$$

In view of (4.2), $u_t(x,t)\geqslant 0$ in $\Omega\times(0,T^*)$ and $p\leqslant 1$, we get

$$w_t-u^p\Delta w-2pu^{p-1}\nabla u\cdot\nabla w\leqslant pu^{-1}u_t w,\quad (x,t)\in\Omega\times\big(0,T^*\big). \tag{4.3}$$

On the other hand, again from (1.1), we know that for $(x,t)\in\partial\Omega\times(0,T^*)$,

$$\begin{aligned} w(x,t) &= u^{-p}(x,t)u_t(x,t)-au(x_0,t) \\ &= u^{-p}(x,t)\int_\Omega g(x,y)u_t(y,t)\,dy-au(x_0,t) \\ &= u^{-p}(x,t)\int_\Omega g(x,y)u^p(y,t)w(y,t)\,dy+\left(u^{-p}(x,t)\int_\Omega g(x,y)u^p(y,t)\,dy-1\right)au(x_0,t) \\ &\leqslant u^{-p}(x,t)\int_\Omega g(x,y)u^p(y,t)w(y,t)\,dy, \end{aligned} \tag{4.4}$$

here we have used the fact that for $(x,t)\in\partial\Omega\times(0,T^*)$,

$$u^{-p}(x,t)\int_\Omega g(x,y)u^p(y,t)\,dy-1\leqslant 0. \tag{4.5}$$

In fact, if $p=1$, (1.1) shows that (4.5) holds. If $p\in(0,1)$, noting $\int_\Omega g(x,y)\,dy<1$ on $\partial\Omega$ and using the Hölder inequality, we get

$$\int_\Omega g(x,y)u^p(y,t)\,dy\leqslant\left(\int_\Omega g(x,y)u(y,t)\,dy\right)^p\left(\int_\Omega g(x,y)\,dy\right)^{1-p}\leqslant\left(\int_\Omega g(x,y)u(y,t)\,dy\right)^p, \tag{4.6}$$

and then (4.5) also follows from (1.1) and the above inequality.

Also the hypothesis (H_4) implies that

$$w(x,0)=\Delta u_0(x)\leqslant 0,\quad x\in\Omega. \tag{4.7}$$

Then from (4.3), (4.4) and (4.7), by using (4.2), (4.5), Lemma 2.2 and Remark 2.3, we know that $w(x,t)=\Delta u(x,t)\leqslant 0$, $(x,t)\in\Omega\times(0,T^*)$. And this complete the proof. □

Lemma 4.2. *Under the same conditions of Lemma* 4.1, *it holds that* $\lim_{t\to T^*}h(t)=\lim_{t\to T^*}H(t)=+\infty$, *and that the blowup set of the solution* $u(x,t)$ *of problem* (1.1) *is the whole domain of* Ω.

Proof. From Lemma 4.1, we have

$$u_t(x,t)\leqslant u^p(x,t)h(t),\quad (x,t)\in\Omega\times\big(0,T^*\big). \tag{4.8}$$

Integrating (4.8) from 0 to t, we get for $x\in\Omega$,

$$\begin{aligned} &\frac{1}{1-p}u^{1-p}(x,t)\leqslant\frac{1}{1-p}u_0^{1-p}(x)+H(t),\quad \text{if } 0<p<1, \\ &\ln u(x,t)\leqslant\ln u_0(x)+H(t),\quad \text{if } p=1. \end{aligned} \tag{4.9}$$

Due to $\lim_{t\to T^*}\sup_{x\in\Omega}u=+\infty$ and $1-p>0$ when $0<p<1$, (4.9) ensures that $\lim_{t\to T^*}H(t)=+\infty$. Since $T^*<+\infty$, from the above equality we have $\lim_{t\to T^*}h(t)=+\infty$.

To show the second conclusion, let $x_1\in\Omega$, $R=\mathrm{dist}(x_1,\partial\Omega)$, $\Omega_1=\{x\colon\ |x-x_1|<R\}$, $r=|x-x_1|$ and consider the following problem

$$\begin{aligned} &v_t=v^p\big(\Delta v+h(t)\big),\quad x\in\Omega_1,\ t>0, \\ &v(x,t)=\eta/2,\quad x\in\partial\Omega_1,\ t>0, \\ &v(x,0)=v_0(x)\leqslant u_0(x),\quad x\in\Omega_1, \end{aligned} \tag{4.10}$$

where η is a positive constant given by (4.2), $v_0(x) > 0$ for $x \in \Omega_1$, $v_0(x) = \eta/2$ for $x \in \partial\Omega_1$, $v_0(x) = v_0(r)$ and $v_0'(r) \leqslant 0$ for $0 \leqslant r \leqslant R$. Then the solution of problem (4.10) exists and satisfies $v(x,t) = v(r,t)$ and $v_r'(r,t) \leqslant 0$ for $0 \leqslant r \leqslant R$, $t \geqslant 0$.

From the proof of Lemma 4.1 we know that the solution $u(x,t)$ of problem (1.1) satisfies $u(x,t) \geqslant \eta > 0$. Then the comparison principle which is a direct consequence of Lemma 2.2 implies that

$$v(x,t) \leqslant u(x,t), \quad x \in \Omega_1,\ t > 0. \tag{4.11}$$

Denote by $\lambda_1 > 0$ and $\varphi(x)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$-\Delta\varphi(x) = \lambda_1\varphi(x), \quad x \in \Omega_1; \qquad \varphi(x) = 0, \quad x \in \partial\Omega_1,$$

such that $\int_{\Omega_1} \varphi(x)\,dx = 1$.

We rewrite Eq. (4.10) as follows

$$v^{-p}v_t = \Delta v + h(t), \quad x \in \Omega_1,\ t > 0. \tag{4.12}$$

Multiplying both sides of (4.12) by $\varphi(x)$ and integrating over $\Omega_1 \times (0,t)$, we get for $t \in (0, T^*)$,

$$\frac{1}{1-p}\int_{\Omega_1} v^{1-p}\varphi\,dx - \frac{1}{1-p}\int_{\Omega_1} v_0^{1-p}\varphi\,dx = \frac{\eta\lambda_1}{2} - \lambda_1\int_0^t\int_{\Omega_1} v\varphi\,dx\,ds + H(t), \quad \text{if } 0 < p < 1,$$

$$\int_{\Omega_1} \varphi\ln v\,dx - \int_{\Omega_1} \varphi\ln v_0\,dx = \frac{\eta\lambda_1}{2} - \lambda_1\int_0^t\int_{\Omega_1} v\varphi\,dx\,ds + H(t), \quad \text{if } p = 1. \tag{4.13}$$

By the above equalities and $\lim_{t\to T^*} H(t) = \infty$ in the first conclusion, we know that if $\int_0^{T^*}\int_{\Omega_1} v\varphi\,dx\,ds < +\infty$ then

$$\limsup_{t\to T^*}\int_{\Omega_1} v^{1-p}\varphi\,dx = +\infty, \quad \text{if } 0 < p < 1,$$

$$\limsup_{t\to T^*}\int_{\Omega_1} \varphi\ln v\,dx = +\infty, \quad \text{if } p = 1. \tag{4.14}$$

Noticing that $v_r'(r,t) \leqslant 0$ for $0 \leqslant r \leqslant R$, $t \geqslant 0$, we have

$$\limsup_{t\to T^*} v(x_1,t) = \infty.$$

On the contrary, if $\int_0^{T^*}\int_{\Omega_1} v\varphi\,dx\,ds = +\infty$, then we also have $\limsup_{t\to T^*} v(x_1,t) = \infty$. In virtue of (4.11) and the arbitrariness of $x_1 \in \Omega$, we obtain the second conclusion. $\square$

Lemma 4.3. *Let hypotheses* (H$_2$)–(H$_4$) *hold, assume that* $\int_\Omega g(x,y)\,dy < 1$ *for* $x \in \partial\Omega$ *and that the solution* $u(x,t)$ *of problem* (1.1) *blows up in finite time* T^**. Then there exists a positive constant d such that*

$$\max_{x\in\overline{\Omega}} u(x,t) \geqslant d\big(T^* - t\big)^{-\frac{1}{p}}, \quad t \in \big(0, T^*\big). \tag{4.15}$$

Proof. We rewrite Eq. (1.1) as follows

$$u^{-p}(x,t)u_t(x,t) = \Delta u(x,t) + au(x_0,t), \quad (x,t) \in \Omega \times \big(0, T^*\big). \tag{4.16}$$

From the definition of $h(t)$ and the first conclusion $\lim_{t\to T^*} h(t) = +\infty$ of Lemma 4.2, we know that $\lim_{t\to T^*} u(x_0,t) = +\infty$. Using the conclusion of Lemma 4.1 and (4.16), we obtain

$$u^{-p}(x_0,t)u_t(x_0,t) \leqslant au(x_0,t), \quad t \in \big(0, T^*\big).$$

Integrating the above inequality over (t, T^*) and noting that $\lim_{t\to T^*} u(x_0,t) = +\infty$, we get

$$u(x_0,t) \geqslant d\big(T^* - t\big)^{-\frac{1}{p}}, \quad t \in \big(0, T^*\big),$$

where $d = (ap)^{-\frac{1}{p}}$. Then we have

$$\max_{x\in\overline{\Omega}} u(x,t) \geqslant u(x_0,t) \geqslant d\big(T^* - t\big)^{-\frac{1}{p}}, \quad t \in \big(0, T^*\big). \qquad \square$$

Lemma 4.4. *Let hypotheses* (H$_2$)–(H$_4$) *hold, assume that* $\int_\Omega g(x,y)\,dy < 1$ *for* $x \in \partial\Omega$ *and that the solution* $u(x,t)$ *of problem* (1.1) *blows up in finite time* T^*. *Then for the case* $0 < p < 1$ *there exists a positive constant* C' *such that*

$$H(t) \leqslant C'\bigl(T^* - t\bigr)^{-\frac{1-p}{p}}, \quad t \in \bigl(0, T^*\bigr). \tag{4.17}$$

Proof. Let $\psi(x)$ be the solution of problem (1.5), then from the proof of Theorem 1.3 we know that there exist two positive constants K_1, K_2 such that $K_2 \leqslant \psi(x) \leqslant K_1$, $x \in \overline{\Omega}$. Set $w(x,t) = A\psi(x)z(t)$, where A is a positive constant to be determined later and $z(t) = H^{\frac{1}{1-p}}(t)$ for the case $0 < p < 1$. From the continuity of the functions $u(x,t)$ and $w(x,t)$ on $\overline{\Omega} \times [0, T^*)$, we know that the set $\{(x,t) \in \overline{\Omega} \times [0,T^*) \mid u(x,t) < w(x,t)\}$ is open, we can denote it by $\Omega_1 \times \bigcup_{i \in S}(t_i, t_{i+1})$, where $\Omega_1 \subset \Omega$ is an open set, $S \subset \mathbb{N}$ is a set of natural integers, $(t_i, t_{i+1}) \subset [0, T^*)$ is an open interval ($i \in S$). Then a direct computation yields for $(x,t) \in \Omega_1 \times (t_i, t_{i+1})$, $i \in S$,

$$\begin{aligned} w_t - w^p\bigl(\Delta w + a w(x_0,t)\bigr) &= A\psi(x)z'(t) - A^{p+1}\psi^p(x)z^{p+1}(t)\bigl(\Delta\psi(x) + a\psi(x_0)\bigr) \\ &\leqslant A\psi^p(x)\bigl[K_1^{1-p}z'(t) - A^p\bigl(a\psi(x_0) - 1\bigr)z^{p+1}(t)\bigr] \\ &= A\psi^p(x)z^p(t)\bigl[K_1^{1-p}h(t)/(1-p) - A^p\bigl(a\psi(x_0) - 1\bigr)z(t)\bigr] \\ &\leqslant A^{p+1}\psi^p(x)z^{p+1}(t)\bigl[a(K_1A)^{1-p}\psi(x_0) - \bigl(a\psi(x_0) - 1\bigr)\bigr]. \end{aligned}$$

If we choose the positive constant A such that $A \leqslant [(a\psi(x_0) - 1)/(aK_1^{1-p}\psi(x_0))]^{\frac{1}{1-p}}$, then from the above inequality we obtain

$$w_t - w^p\bigl(\Delta w + a w(x_0,t)\bigr) \leqslant 0, \quad (x,t) \in \Omega_1 \times (t_i, t_{i+1}),\ i \in S.$$

On the other hand, for $(x,t) \in \partial\Omega_1 \times (t_i, t_{i+1}) \cup \Omega_1 \times \{t_i\}$, $i \in S$, $w(x,t) = u(x,t)$. All these inequalities and the comparison principle which is a direct consequence of Lemma 2.2 implies that $w(x,t) \leqslant u(x,t)$ on $\overline{\Omega}_1 \times [t_i, t_{i+1})$, $i \in S$, and then the set $\{(x,t) \in \overline{\Omega} \times [0,T^*) \mid u(x,t) < w(x,t)\}$ is empty, that is to say, $u(x,t) \geqslant w(x,t)$, $(x,t) \in \overline{\Omega} \times [0,T^*)$. Then we have,

$$h(t) \geqslant aA\psi(x_0)z(t). \tag{4.18}$$

In virtue of the definition of $z(t)$, by integrating the above inequality over (t, T^*), we obtain the inequalities in (4.17), where $C' = [paA\psi(x_0)/(1-p)]^{-\frac{1-p}{p}}$. $\square$

Lemma 4.5. *Let hypotheses* (H$_2$)–(H$_4$) *hold, assume that* $\int_\Omega g(x,y)\,dy < 1$ *for* $x \in \partial\Omega$ *and that the solution* $u(x,t)$ *of problem* (1.1) *blows up in finite time* T^*. *Then there exist two positive constants* D *and* D' *such that*

$$\max_{x \in \overline{\Omega}} u(x,t) \leqslant D\bigl(T^* - t\bigr)^{-\frac{1}{p}} + D', \quad t \in \bigl(0, T^*\bigr), \tag{4.19}$$

where $D = [2(1-p)C']^{\frac{1}{1-p}}$ *and* $D' = 2^{\frac{1}{1-p}}\max_{x \in \overline{\Omega}} u_0(x)$.

Proof. Denote $U(t) = \max_{x \in \overline{\Omega}} u(x,t)$. Using Eq. (1.1) and Theorem 4.5 in [14], we get

$$U'(t) \leqslant U^p(t)h(t), \quad \text{a.e. } t \in \bigl(0, T^*\bigr).$$

Rewrite it as follows

$$U^{-p}(t)U'(t) \leqslant h(t), \quad \text{a.e. } t \in \bigl(0, T^*\bigr)$$

and integrate it over $(0,t)$, we have

$$\frac{1}{1-p}U^{1-p}(t) - \frac{1}{1-p}U^{1-p}(0) \leqslant H(t), \quad t \in \bigl(0, T^*\bigr).$$

In virtue of the conclusion (4.17) of Lemma 4.4, we get the desired result. $\square$

From Lemmas 4.1–4.5, we complete the proof of Theorem 1.5.

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